Identifying two-photon high-dimensional entanglement in transverse patterns

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We propose a scheme to explore two-photon high-dimensional entanglement associated with a transverse pattern by means of two-photon interference in a beamsplitter. We find that the topological symmetry of the angular spectrum of the two-photon state governs the nature of the two-photon interference. We prove that the anti-coalescence interference is the signature of two-photon entanglement. On the basis of this feature, we propose a special Mach-Zehnder interferometer incorporated with two spiral phase plates which can change the interference from a coalescence to an anti-coalescence type only for a two-photon entangled state. The scheme is simple and straightforward compared with the test for a Bell inequality.

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Quantum entanglement in a high-dimensional Hilbert space have potential applications in quantum cryptography with higher alphabets and in quantum communication with increased information flux. Recently, studies on the conservation and entanglement of orbital angular momentum (OAM) of photons generated by spontaneous parametric down-conversion (SPDC) have drawn much attention [1]- [11]. This provides the possibility of realizing photon entanglement in higher dimensions. To demonstrate the ultimate evidence of OAM entanglement, a generalized Bell inequality should be taken into account [3,4]. In this work we propose a simple scheme to identify the two-photon entanglement associated with transverse patterns by means of two-photon interference in a beamsplitter. The two-photon interference in a beamsplitter, first reported experimentally in Refs. [12,13], is regarded as an effective way to test quantum nonlocality. In Refs. [14,15] the concept was put forward that the topological symmetry of the probability-amplitude spectrum of a two-photon wavepacket in the frequency domain determines the nature of the two-photon interference, which may appear as coalescence interference (CI) or anti-coalescence interference (ACI) according to the decrease or increase of the coincidence probability in the absence of interference, respectively. The key feature is that the ACI effect is the signature of two-photon entanglement. At the same time, Walborn et al [6] showed that in the SPDC process the spatial symmetry of the pump beam is related to the interference behavior of two down-converted photons. Not limited to the SPDC case, we survey this issue for a general two-photon state associated with transverse patterns. We prove that ACI is evidence of two-photon entanglement. To reveal the two-photon entanglement, we propose a special Mach-Zehnder interferometer incorporated with two spiral phase plates (SPP), which may change the topological symmetry of a two-photon entangled angular spectrum. This scheme can thus be utilized to explore two-photon high-dimensional entanglement in a transverse pattern.

To begin with, we consider a 50/50 lossless beamsplitter which satisfies a transform relation between two input beams $a_j(q_x, q_y)$ and two output beams $b_j(q_x, q_y)$

$$\begin{aligned} b_1(q_x,q_y) &= (1/\sqrt{2})[a_1(q_x,q_y)e^{i\varphi_\tau} + a_2(q_x,-q_y)e^{i\varphi_\rho}], \\ b_2(q_x,q_y) &= (1/\sqrt{2})[-a_1(q_x,-q_y)e^{-i\varphi_\rho} + a_2(q_x,q_y)e^{-i\varphi_\tau}], \end{aligned} \tag{1}$$

where φ_τ and φ_ρ are two phases allowed in the unitary transform, and q_x and q_y are two orthogonally transverse wavevectors defined in right-handed coordinates, as shown in Fig. 1. The negative sign before component q_y refers to reflection of the beam if q_y is in the incident plane [6]. According to Ref. [14], for an input two-photon state $|\Phi\rangle_{in} = F(a_1^\dagger, a_2^\dagger)|0\rangle$, the corresponding output state is obtained by $|\Phi\rangle_{out} = F(\overline{b}_1^\dagger, \overline{b}_2^\dagger)|0\rangle$, where $\overline{b}_1^\dagger$ and $\overline{b}_2^\dagger$ are written as

$$\begin{aligned} \overline{b}_1^\dagger(q_x,q_y) &= (1/\sqrt{2})[a_1^\dagger(q_x,q_y)e^{i\varphi_\tau} - a_2^\dagger(q_x,-q_y)e^{-i\varphi_\rho}], \\ \overline{b}_2^\dagger(q_x,q_y) &= (1/\sqrt{2})[a_1^\dagger(q_x,-q_y)e^{i\varphi_\rho} + a_2^\dagger(q_x,q_y)e^{-i\varphi_\tau}]. \end{aligned} \tag{2}$$

For simplicity, we assume that the two photons are monochromatic and degenerate in both polarization and frequency, but propagate in different directions. A general two-photon state associated with the transverse field is described as

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$$|\Phi\rangle_{in} = \int d\mathbf{q}_1 d\mathbf{q}_2 \Phi(\mathbf{q}_1,\mathbf{q}_2) a_1^\dagger(\mathbf{q}_1) a_2^\dagger(\mathbf{q}_2)|0\rangle = \int d\mathbf{x}_1 d\mathbf{x}_2 \Phi(\mathbf{x}_1,\mathbf{x}_2) a_1^\dagger(\mathbf{x}_1) a_2^\dagger(\mathbf{x}_2)|0\rangle, \tag{3}$$

where $\mathbf{q}_j = (q_{xj}, q_{yj})$ is the transverse wavevector for photon j, and $\Phi(\mathbf{q}_1,\mathbf{q}_2)$ is the two-photon angular spectrum. The state can also be expressed with the transverse positions by using Fourier transformations $\Phi(\mathbf{x}_1,\mathbf{x}_2) = \mathcal{F}^{-1}[\Phi(\mathbf{q}_1,\mathbf{q}_2)] \equiv (2\pi)^{-2}\int d\mathbf{q}_1 d\mathbf{q}_2 \Phi(\mathbf{q}_1,\mathbf{q}_2)\exp[i(\mathbf{x}_1\cdot\mathbf{q}_1 + \mathbf{x}_2\cdot\mathbf{q}_2)]$ and $a_j(\mathbf{x}_j) = \mathcal{F}^{-1}[a_j(\mathbf{q}_j)]$. Consider state (3) as the input of the beamsplitter, then the outgoing state is obtained to be

$$\begin{aligned} |\Phi\rangle_{out} = (1/2)\int d\mathbf{q}_1 d\mathbf{q}_2 [\Phi(q_{x1}, q_{y1}, q_{x2}, -q_{y2}) e^{i\varphi} a_1^\dagger(\mathbf{q}_1) a_1^\dagger(\mathbf{q}_2) - \Phi(q_{x1}, -q_{y1}, q_{x2}, q_{y2}) e^{-i\varphi} a_2^\dagger(\mathbf{q}_1) a_2^\dagger(\mathbf{q}_2)]|0\rangle \\ +(1/2)\int d\mathbf{q}_1 d\mathbf{q}_2 [\Phi(q_{x1}, q_{y1}, q_{x2}, q_{y2}) - \Phi(q_{x2}, -q_{y2}, q_{x1}, -q_{y1})] a_1^\dagger(\mathbf{q}_1) a_2^\dagger(\mathbf{q}_2)]|0\rangle, \end{aligned} \tag{4}$$

where $\varphi = \varphi_\tau + \varphi_\rho$. The coincidence probability in the two outgoing arms is evaluated as

$$P_c = (1/2)[1 - \int d\mathbf{q}_1 d\mathbf{q}_2 \Phi(q_{x1}, q_{y1}, q_{x2}, q_{y2}) \Phi^*(q_{x2}, -q_{y2}, q_{x1}, -q_{y1})]. \tag{5}$$

When the two-photon angular spectrum satisfies the symmetric condition, $\Phi(q_{x1}, q_{y1}, q_{x2}, q_{y2}) = \Phi(q_{x2}, -q_{y2}, q_{x1}, -q_{y1})$, the coincidence probability is null and perfect CI occurs. Contrarily, when the spectrum satisfies the anti-symmetric condition $\Phi(q_{x1}, q_{y1}, q_{x2}, q_{y2}) = -\Phi(q_{x2}, -q_{y2}, q_{x1}, -q_{y1})$, the coincidence probability is unity and perfect ACI occurs. In the latter case, we can prove that $|\Phi\rangle_{out} = |\Phi\rangle_{in}$, that is, the beamsplitter becomes transparent due to the quantum interference [14]. In polar coordinates (q,θ), the conditions are written as $\Phi(q_1, q_2, \theta_1, \theta_2) = \pm\Phi(q_2, q_1, 2\pi - \theta_2, 2\pi - \theta_1)$.

For a non-entangled two-photon state, $\Phi(\mathbf{q}_1,\mathbf{q}_2) = \Phi_1(\mathbf{q}_1)\Phi_2(\mathbf{q}_2)$, we calculate the integral in Eq. (5) $\int d\mathbf{q}_1 d\mathbf{q}_2 \Phi(q_{x1}, q_{y1}, q_{x2}, q_{y2}) \Phi^*(q_{x2}, -q_{y2}, q_{x1}, -q_{y1}) = |\int dq_x dq_y \Phi_1(q_x, q_y)\Phi_2^*(q_x, -q_y)|^2 \geq 0$, so that ACI never happens ($P_c \leq 1/2$). In other words, we can identify two-photon entanglement by the condition $P_c > 1/2$. We note that the above discussion is also valid in the position space after replacing $\mathbf{q}_j$ by $\mathbf{x}_j$.

We now show some examples. The first example seems to contradict our common knowledge of two-photon interference. Two independent degenerate photons with the same OAM $|l,l\rangle$ do not interfere in a beamsplitter ($P_c = 1/2$), whereas two photons with opposite OAM $|l,-l\rangle$ show perfect CI ($P_c = 0$). This is because the transmission does not change a photon's OAM while the reflection results in an opposite OAM. Then we consider a set of Bell states involving the photon's OAM

$$|\Psi^\pm\rangle = (1/\sqrt{2})(|l,l\rangle \pm |-l,-l\rangle) = \int d\mathbf{q}_1 d\mathbf{q}_2 R(q_1) R(q_2)(e^{il(\theta_1+\theta_2)} \pm e^{-il(\theta_1+\theta_2)}) a_1^\dagger(\mathbf{q}_1) a_2^\dagger(\mathbf{q}_2)|0\rangle, \tag{6a}$$

$$|\Phi^\pm\rangle = (1/\sqrt{2})(|l,-l\rangle \pm |-l,l\rangle) = \int d\mathbf{q}_1 d\mathbf{q}_2 R(q_1) R(q_2)(e^{il(\theta_1-\theta_2)} \pm e^{-il(\theta_1-\theta_2)}) a_1^\dagger(\mathbf{q}_1) a_2^\dagger(\mathbf{q}_2)|0\rangle. \tag{6b}$$

It is readily seen that $|\Psi^+\rangle$ and $|\Phi^\pm\rangle$ satisfy the symmetric condition while $|\Psi^-\rangle$ satisfies the anti-symmetric one. This feature is similar to the Bell basis associated with polarization and frequency [17]. However, for the OAM Bell bases, two of them can be gained from the other two through the reflection of the two photons, i.e. $|\Psi^\pm\rangle \leftrightarrow |\Phi^\pm\rangle$.

In the SPDC process, the signal and idler photon pair can be described as a two-photon state

$$|\Phi\rangle = \frac{1}{\pi}\sqrt{\frac{2L}{k_p}}\int d\mathbf{q}_1 d\mathbf{q}_2 \upsilon(\mathbf{q}_1 + \mathbf{q}_2)\mathrm{sinc}[L|\mathbf{q}_1 - \mathbf{q}_2|^2/(4k_p)] a_1^\dagger(\mathbf{q}_1) a_2^\dagger(\mathbf{q}_2)|0\rangle, \tag{7}$$

where $\upsilon(\mathbf{q})$ is the angular spectrum of the pump beam with the wavenumber k_p and L is the length of the crystal. In Eq. (7), the sinc function satisfies the symmetric condition, so that the symmetry of $\upsilon(\mathbf{q})$ dominates the nature of the two-photon interference. When the pump beam is in the Gaussian fundamental mode with a beam waist w_0, $\upsilon(\mathbf{q}) = G_{00}(\mathbf{q}) = (w_0/\sqrt{2\pi})\exp(-|\mathbf{q}|^2 w_0^2/4)$, $\upsilon(\mathbf{q}_1 + \mathbf{q}_2)$ satisfies the symmetric condition. If the mode structure of the pump beam is Hermitian-Gaussian $HG_{mn}(\mathbf{q})$, then symmetric and anti-symmetric two-photon spectra are obtained for even and odd numbers of the subscript n, respectively. This result has been demonstrated experimentally in Fig. 3 of Ref. [6], where CI and ACI occur for the pump modes HG_{10} and HG_{01}, respectively, in the case of the symmetric polarization.

To explore two-photon entanglement for an angular spectrum satisfying the symmetric condition, we propose a scheme which may change the symmetry of an entangled two-photon spectrum without introducing any additional entanglement. The key device in the scheme, shown in Fig. 2, is a Mach-Zehnder interferometer (MZI) containing two identical spiral phase plates (SPPs). The role of the latter is to produce a phase shift $\exp(i\varsigma\theta)$ linearly distributed along

the azimuthal angle θ [4]. Because the two SPPs are the same, there is coherent superposition of the phases $\exp(i\varsigma\theta)$ and $\exp(-i\varsigma\theta)$, due to transmission and reflection, respectively, in each outgoing arm of the MZI. An adjustable phase shifter (PS) is inserted into the path of beam a_1. Note that the two beams are reflected twice by the mirrors before the last beamsplitter BS, so that the spectrum function is invariant. An angular spectrum $\Phi(\mathbf{q}_1, \mathbf{q}_2)$ of a two-photon state is defined at the source. In the paraxial approximation, a beam propagating a distance z will introduce a phase $\exp[ikz - iq^2 z/(2k)]$. While beam a_2 freely propagates a distance z_2, beam a_1 is divided into two beams by beamsplitter BS1 and travels a distance z_1 to the SPP, thus the two-photon state can be written as

$$|\Phi\rangle_1 = (1/\sqrt{2})e^{ik(z_1+z_2)} \int d\mathbf{q}_1 d\mathbf{q}_2 e^{-i(q_1^2 z_1 + q_2^2 z_2)/(2k)} [\Phi(\mathbf{q}_1, \mathbf{q}_2) e^{i\varphi_{1\tau}} a_1^\dagger(\mathbf{q}_1) - \Phi(q_{x1}, -q_{y1}, \mathbf{q}_2) e^{-i\varphi_{1\rho}} a_3^\dagger(\mathbf{q}_1)] a_2^\dagger(\mathbf{q}_2)|0\rangle, \tag{8}$$

where $\varphi_{j\tau}$ and $\varphi_{j\rho}$ are the phases associated with BS-j (j=1,2). The state can also be expressed with the position variables. Considering the phases $\exp(i\varsigma\theta)$ and $\exp(i\varphi)$ introduced by the SPPs and the phase shifter, respectively, we obtain

$$|\Phi\rangle_2 = (1/\sqrt{2}) \int d\mathbf{x}_1 d\mathbf{x}_2 [\Psi(\mathbf{x}_1, \mathbf{x}_2, z_1, z_2) e^{i(\varphi_{1\tau}+\varphi)} a_1^\dagger(\mathbf{x}_1) - \Psi(x_1, -y_1, \mathbf{x}_2, z_1, z_2) e^{-i\varphi_{1\rho}} a_3^\dagger(\mathbf{x}_1)] e^{i\varsigma\theta_1} a_2^\dagger(\mathbf{x}_2)|0\rangle,$$

where $\Psi(\mathbf{x}_1, \mathbf{x}_2, z_1, z_2) = \mathcal{F}^{-1}[\Phi(\mathbf{q}_1, \mathbf{q}_2) e^{ik(z_1+z_2) - i(q_1^2 z_1 + q_2^2 z_2)/(2k)}]$. After passing through BS2, the two-photon state as it arrives at the last BS is given by

$$|\Phi\rangle_3 = ie^{i(\varsigma\pi + \alpha_-)} \int d\mathbf{x}_1 d\mathbf{x}_2 \Psi(\mathbf{x}_1, \mathbf{x}_2, z_1, z_2) \sin[\varsigma(\theta_1 - \pi) + \alpha_+] a_1^\dagger(\mathbf{x}_1) a_2^\dagger(\mathbf{x}_2)|0\rangle, \tag{9}$$

where $\alpha_\pm = (1/2)[\varphi + \varphi_{1\tau} + \varphi_{2\tau} \pm (\varphi_{1\rho} - \varphi_{2\rho})]$. Beam a_3 has been omitted here since it is not involved in the interference in the last BS. For simplicity, we assume that the SPPs are close to the last BS and hence the propagation distance between the SPPs and the last BS can be neglected [16].

Equation (9) is compatible with that of the similar scheme for two-photon entanglement in the frequency domain, which was experimentally implemented [18,19] and later theoretically explained [17]. As an example, we consider a two-photon entangled state generated by SPDC with a thin crystal where the sinc function is not taken into account. When the Gaussian mode G_{00} pumps the crystal, we obtain

$$\Psi(\mathbf{x}_1, \mathbf{x}_2, z, z) = A \exp\left\{ -\frac{|\mathbf{x}_1 + \mathbf{x}_2|^2}{4w^2(z)} + i\frac{k_p}{4}\left[\frac{z_0^2 |\mathbf{x}_1 - \mathbf{x}_2|^2}{2z^2 R(z)} + \frac{|\mathbf{x}_1|^2 + |\mathbf{x}_2|^2}{R(z)} \right] \right\}, \tag{10}$$

where $k_p = 2k$ is the wavenumber of the pump beam, $z_0 = k_p w_0^2/2$ the Rayleigh length, $w(z) = w_0(1 + z^2/z_0^2)^{1/2}$ the spot size and $R(z) = (z^2 + z_0^2)/z$ the radius of curvature of the Gaussian beam. According to Eq. (5), the phase part in Eq. (10) does not contribute to the coincidence probability. In Fig. 3a, we show the numerical result of the coincidence probability as a function of the parameter ς using Eqs. (5), (9) and (10). For $\alpha_+ = 0$ ($\alpha_+ = \pi/2$), the odd and even (even and odd) numbers ς contribute to maximum CI and ACI effects, respectively. This result can be explained by the symmetry of the wavefunction. When the aperture of the optical system is much larger than the spot size $w(z)$, the Gaussian function in Eq. (10) is closer to $\delta(\mathbf{x}_1 + \mathbf{x}_2)$, resulting in $\theta_1 = \theta_2 + \pi$. Relying on the strong correlation between θ_1 and θ_2 of the two photons, the function $\sin[\varsigma(\theta_1 - \pi)]$ can satisfy either the symmetric or anti-symmetric conditions provided ς is taken as odd or even, respectively. Figure 3b shows the coincidence probability versus the phase α_+ for $\varsigma = 1$ and 2. Experimentally, the reference for the coincidence probability can be acquired by setting a path difference for the two photons so that interference does not occur in the beamsplitter. The coincidence probability larger than this reference witnesses two-photon entanglement.

In summary, we have demonstrated that the nature of two-photon interference in a beamsplitter is related to the topological symmetry of the two-photon angular spectrum. The ACI effect is the signature of two-photon entanglement. On the basis of this feature, we have proposed a scheme which can identify two-photon high-dimensional entanglement in transverse patterns by means of a special Mach-Zehnder interferometer.

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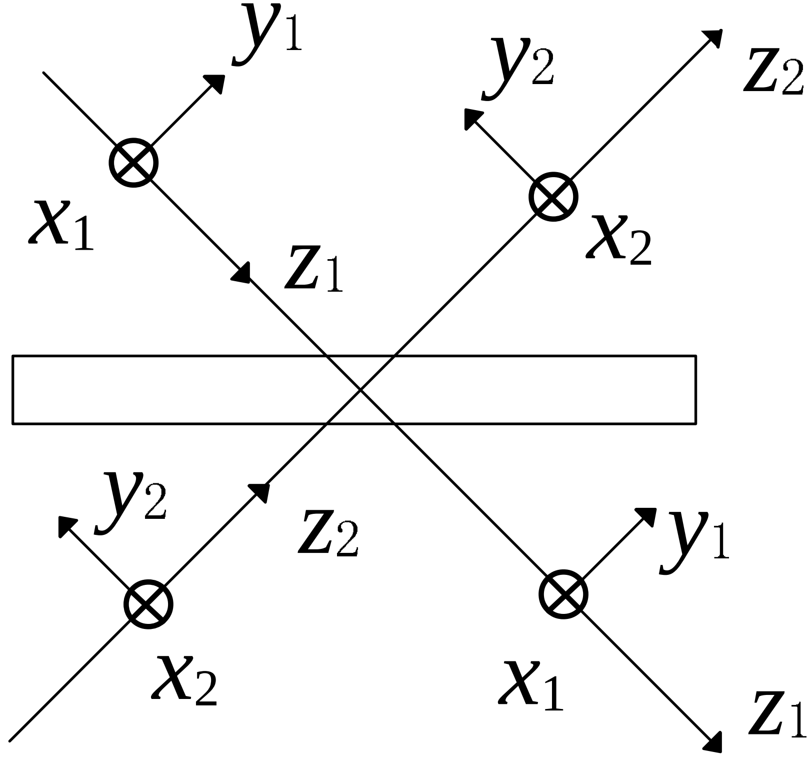
Figure Captions

Fig. 1. Sketch of two-photon interference in a beamsplitter. Right handed coordinates are used for each beam.

Fig. 2. A scheme to explore two-photon entanglement in transverse patterns. Beam a_1 passes through a Mach-Zehnder interferometer (MZI) consisting of two beamsplitters (BS1 and BS2) and two mirrors. Two identical spiral phase plates (SPP) are put in each path of the MZI and an adjustable phase shifter (PS) is in the path of beam a_1. After MZI, beams a_1 and a_2 interfere in the last beamsplitter (BS) and coincidence measurement is performed for the two outgoing beams.

Fig. 3. (a) Coincidence probability as a function of the SPP parameter ς for $\alpha_+ = 0$ and $\pi/2$; (b) Coincidence probability as a function of the phase α_+ for the SPP parameter $\varsigma = 1$ and 2. In the numerical simulation, the aperture of the optical system is 40 times larger than the spot size $w(z)$.



y_1
y_2
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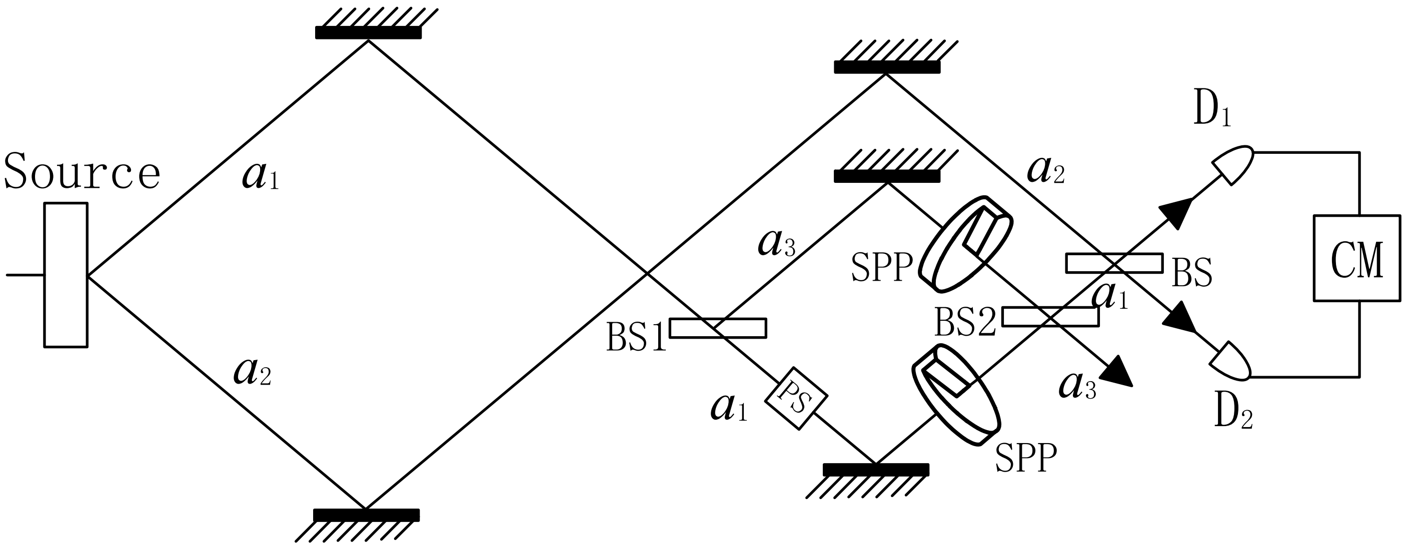
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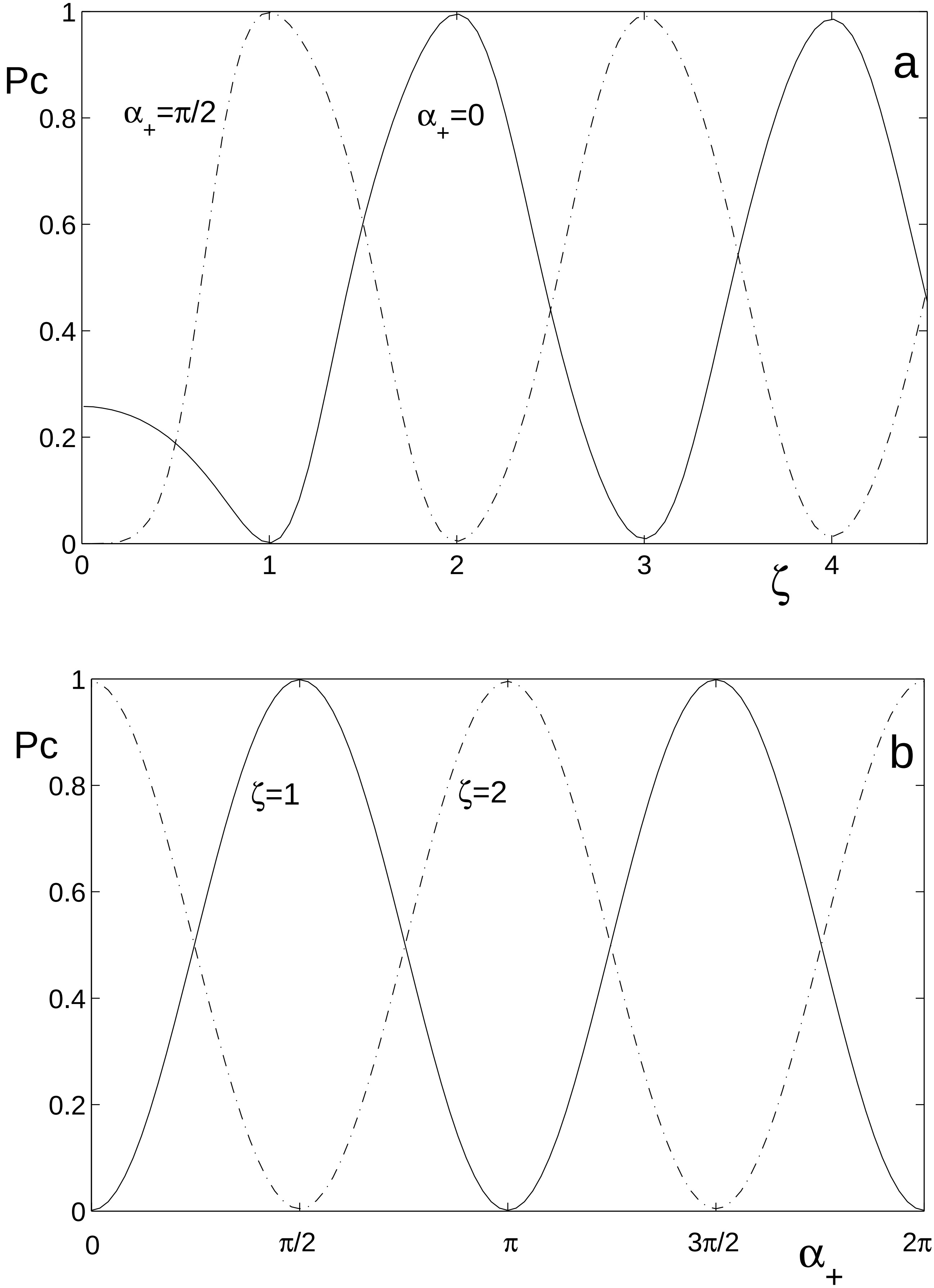

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Pc
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α+=0
ζ
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0.2
0.4
0.6
0.8
1
0
1
2
3
4
b
Pc
ζ=1
ζ=2
α+
0
π/2
π
3π/2
2π